

# Some mathematical aspects in determining the 3D controlled solutions of the Gross-Pitaevskii equation

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## Abstract

The possibility of the decomposition of the three dimensional (3D) Gross-Pitaevskii equation (GPE) into a pair of coupled Schrödinger-type equations, is investigated. It is shown that, under suitable mathematical conditions, solutions of the 3D controlled GPE can be constructed from the solutions of a 2D linear Schrödinger equation (transverse component of the GPE) coupled with a 1D nonlinear Schrödinger equation (longitudinal component of the GPE). Such a decomposition, called the 'controlling potential method' (CPM), allows one to cast the above solutions in the form of the product of the solutions of the transverse and the longitudinal components of the GPE. The coupling between these two equations is the functional of both the transverse and the longitudinal profiles. The analysis shows that the CPM is based on the variational principle that sets up a condition on the controlling potential well, and whose physical interpretation is given in terms of the minimization of the (energy) effects introduced by the control operation.

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## I. INTRODUCTION

Since its experimental observation [1], the study of the three dimensional (3D) collective and nonlinear dynamics of the Bose Einstein condensate (BEC) [2] in an external potential trap [3] has received a great deal of attention by a very wide scientific community and in the investigations concerning fundamental physics, by mathematical physics and sophisticated technological applications [4]. Although rapid scientific and technological advances have been achieved in this area, finding the exact analytical 3D solutions of the Gross-Pitaevskii equation (GPE) [3], that correspond to the coherent state of a BEC in a suitable external potential well (such as soliton-like structures), still remains a challenging task for physicists and mathematicians.

A number of valuable approximative analytical [5] and numerical evaluations [6] have been presented in the literature and have been adequately compared with a very wide spectrum of experimental observations. The experience gained from these investigations may suggest the idea that a BEC's dynamics exhibits the features of a nonlinear non-autonomous system [7] for which it seems to be necessary to include some control operations in order to allow the existence of coherent structures. In particular, to retain the 3D coherent stationary structures of the BEC for a long time, suitable "ad hoc" time-dependent external potentials and control operations are known to be necessary [8]. Furthermore, in the presence of an inhomogeneous time-dependent external potential one encounters some difficulties to find exact soliton solutions in one or more dimensions, although several kind of solitons have been found in certain approximations [9]. Consequently, one easily arrives to the conclusion that, in order to get exact soliton structures, some sort of the 'control of the system' seems to be necessary. This implies that the correct analysis of the system should include a control potential term in the GPE which is to be determined dynamically by the system itself. In principle, this procedure may be extended to an arbitrary 'controlled solution' with the appropriate choice of the external potential (so-called 'controlling potential' [10]). In fact, a controlling potential method (CPM) has been recently proposed in the literature and used to find multi-dimensional controlled localized solutions of the GPE. In the preliminary investigations [11], this method has established reasonable experimental control operations that ensure the stability of the solution against relatively small errors in the experimental realization of the prescribed controlling potential. The main goal of the CPM is to fix the

type of the desired controlled solution and to find the appropriate family of the controlling potentials. Then, the set of suitable mathematical conditions has to be found allowing us to select the desired solution, with the adopted controlling potential.

In this manuscript, we develop an analytical procedure to construct exact three dimensional solutions of a controlled Gross-Pitaevskii equation, by improving the CPM. To this end, we develop the theory of the BEC control based on two *decomposition theorems* leading to suitable physical conditions to express the BEC wave function as the product of a 2D wave function and a 1D wave function, taking into account the 'transverse' and 'longitudinal' BEC profiles, respectively. Such a factorization allows us to decompose the 3D controlled GPE into a set of coupled equations, comprising a 2D linear Schrödinger equation (governing the evolution of the 'transverse' wave function), a 1D nonlinear Schrödinger equation (governing the evolution of the 'longitudinal' wave function) and a variational condition involving the controlling potential. The requirement for the minimization of the effects introduced by the control operations (i. e. the requirement that the average of the controlling potential over the transverse plane is equal to zero) allows us to determine explicitly the self-consistent controlling potential which also plays the role of the coupling term between transverse and longitudinal BEC dynamics.

## II. CONTROLLED GROSS-PITAEVSKII EQUATION

It is well known that the spatio-temporal evolution of the ultracold system of identical atoms forming a BEC in the presence of the external potential  $U_{ext}(\mathbf{r}, t)$ , within the mean field approximation, is governed by the three dimensional Gross-Pitaevskii equation [3], viz.,

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m_a} \nabla^2 \Psi + NQ |\Psi|^2 \Psi + U_{ext}(\mathbf{r}, t) \Psi, \quad (1)$$

where  $\Psi(\mathbf{r}, t)$  is the wavefunction describing the BEC state,  $m_a$  is the atom mass and  $Q$  is a coupling coefficient related to the short range scattering (s-wave) length  $a$  representing the interactions between atomic particles, viz.,  $Q = 4\pi\hbar^2 a/m_a$ , and  $N$  is the number of atoms. Note that the short range scattering length can be either positive or negative. We assume that  $U_{ext}$  is the sum of the 3D trapping potential well,  $U_{trap}$ , that is used to confine the particles of a BEC, and the controlling potential  $U_{contr}$  which will be determined self-consistently. We conveniently introduce the variable  $s = ct$  ( $c$  being the speed of light) and

divide both sides of Eq.(1) by  $m_a c^2$ , and we use the notation

$$\frac{U_{ext}(\mathbf{r}, t)}{m_a c^2} = \frac{U_{trap}(\mathbf{r}, t)}{m_a c^2} + \frac{U_{contr}(\mathbf{r}, t)}{m_a c^2} \equiv V_{trap}(\mathbf{r}, s) + V_{contr}(\mathbf{r}, s), \quad (2)$$

Eq. (1) can be cast in the form

$$i \lambda_c \frac{\partial \psi}{\partial s} = -\frac{\lambda_c^2}{2} \nabla^2 \psi + [V_{trap}(\mathbf{r}, s) + V_{contr}(\mathbf{r}, s) + q|\psi|^2] \psi, \quad (3)$$

where  $\psi(\mathbf{r}, s) \equiv \Psi(\mathbf{r}, t = s/c)$ ,  $\lambda_c \equiv \hbar/m_a c^2$  is the Compton wavelength of the single atom of BEC and  $q \equiv NQ/mc^2$ .

In this paper, we will investigate the properties of Eq. (3) and  $V_{contr}$  that enable the existence of the controlled 3D solutions in the factorized form

$$\psi(\mathbf{r}, s) = \psi_{\perp}(\mathbf{r}_{\perp}, s) \psi_z(\mathbf{r}_{\perp}, z, s), \quad (4)$$

provided that  $V_{trap}$  can be split into two parts, as

$$V_{trap}(\mathbf{r}, s) = V_{\perp}(\mathbf{r}_{\perp}, s) + V_z(z, s) \quad (5)$$

where, in Cartesian coordinates,  $\mathbf{r} \equiv (x, y, z)$  and  $\mathbf{r}_{\perp} \equiv (x, y)$  denotes, by definition, the 'transverse' part of the particle's vector position  $\mathbf{r}$ . We also refer to  $z$  as to the 'longitudinal' coordinate.

By substituting Eqs. (4) and (5) in Eq. (3), we easily get:

$$\begin{aligned} \psi_{\perp} \frac{\lambda_c^2}{2} \left[ \nabla_{\perp}^2 \psi_z + 2 \frac{\nabla_{\perp} \psi_{\perp}}{\psi_{\perp}} \cdot \nabla_{\perp} \psi_z \right] + \psi_{\perp} \left[ i \lambda_c \frac{\partial \psi_z}{\partial s} + \frac{\lambda_c^2}{2} \frac{\partial^2 \psi_z}{\partial z^2} - (V_z + V_{contr} + q|\psi_{\perp}|^2 |\psi_z|^2) \psi_z \right] \\ + \psi_z \left[ i \lambda_c \frac{\partial \psi_{\perp}}{\partial s} + \frac{\lambda_c^2}{2} \nabla_{\perp}^2 \psi_{\perp} - V_{\perp}(\mathbf{r}_{\perp}, s) \psi_{\perp} \right] = 0, \end{aligned} \quad (6)$$

where, in Cartesian coordinates,  $\nabla_{\perp} \equiv \hat{x} \partial / \partial x + \hat{y} \partial / \partial y$ .

Let us define as 'controlled parameter' the following time-dependent quantity:

$$q_{1D}(s) = q \int d^2 \vec{r}_{\perp} |\psi_{\perp}|^4; \quad (7)$$

and the following linear and nonlinear operators, respectively:

$$\hat{H}_{\perp} = -\frac{\lambda_c^2}{2} \nabla_{\perp}^2 + V_{\perp}(\mathbf{r}_{\perp}, s) \quad (8)$$

$$\hat{H}_z = -\frac{\lambda_c^2}{2} \frac{\partial^2}{\partial z^2} + V_z(z, s) + q_{1D}(s) |\psi_z(\mathbf{r}_{\perp}, z, s)|^2 + V_0 \quad (9)$$

where  $V_0$  is an arbitrary real constant. Then, Eq. (6) can be rewritten as:

$$\begin{aligned} \psi_z \left( i \lambda_c \frac{\partial}{\partial s} - \hat{H}_{\perp} \right) \psi_{\perp} + \psi_{\perp} \left[ \left( i \lambda_c \frac{\partial}{\partial s} - \hat{H}_z \right) \psi_z + (q_{1D}(s) - q|\psi_{\perp}|^2) |\psi_z|^2 \psi_z + (V_0 - V_{contr}) \psi_z \right] \\ + \psi_{\perp} \frac{\lambda_c^2}{2} \left[ \nabla_{\perp}^2 \psi_z + 2 \frac{\nabla_{\perp} \psi_{\perp}}{\psi_{\perp}} \cdot \nabla_{\perp} \psi_z \right] = 0. \end{aligned} \quad (10)$$

### III. THE DECOMPOSITION PROPERTIES OF THE CONTROLLED GROSS-PITAIEVSKII EQUATION

By the definition of the controlling potential,  $V_{contr}$  depends both on  $\psi_\perp$  and  $\psi_z$ . In particular, we assume here that the space and time dependence of  $V_{contr}$  is given also through  $\rho_\perp(\mathbf{r}_\perp, s) \equiv |\psi_\perp(\mathbf{r}_\perp, s)|^2$ , viz.,

$$V_{contr} = V_{contr}(\rho_\perp(\mathbf{r}_\perp, s), z, s). \quad (11)$$

Moreover, defining also the following functional of  $\rho_\perp$ :

$$\mathcal{V}[\rho_\perp; z, s] = \int \rho_\perp(\mathbf{r}_\perp, s) V_{contr}(\rho_\perp(\mathbf{r}_\perp, s), z, s) d^2\mathbf{r}_\perp, \quad (12)$$

the following theorem holds:

#### DECOMPOSITION THEOREM 1.

If

$$\psi_z(\mathbf{r}_\perp, z, s) = \psi_z(z, s), \quad (13)$$

and  $\psi_\perp(\mathbf{r}_\perp, s)$  is the solution of the following 2D linear Schrödinger equation

$$\left( i \lambda_c \frac{\partial}{\partial s} - \hat{H}_\perp \right) \psi_\perp = 0, \quad (14)$$

and  $\mathcal{V}$  is a stationary functional (with respect to variations  $\delta\rho_\perp$  of  $\rho_\perp$ ), assuming the value  $\mathcal{V} = V_0$ , conditioned by the constraints

$$\int \rho_\perp d^2\mathbf{r}_\perp = 1, \quad (15)$$

(normalization condition for  $\psi_\perp$ ), and

$$\int \rho_\perp^2 d^2\mathbf{r}_\perp = \frac{q_{1D}(s)}{q} = \text{given function}, \quad (16)$$

then  $\psi_z$  is the solution of the following 1D nonlinear Schrödinger equation

$$\left( i \lambda_c \frac{\partial}{\partial s} - \hat{H}_z \right) \psi_z = 0, \quad (17)$$

and  $V_{contr}$  is given by

$$V_{contr}(\mathbf{r}_\perp, z, s) = [q_{1D}(s) - q|\psi_\perp(\mathbf{r}_\perp, s)|^2] |\psi_z(z, s)|^2 + V_0. \quad (18)$$

To prove this theorem, first of all, we note that the assumptions (13) and (14) allow us to reduce Eq. (10) to

$$\left(i \lambda_c \frac{\partial}{\partial s} - \hat{H}_z\right) \psi_z + [q_{1D}(s) - q|\psi_\perp|^2] |\psi_z|^2 \psi_z + (V_0 - V_{contr}) \psi_z = 0. \quad (19)$$

Secondly, the required stationarity of  $\mathcal{V}$  with respect to variations  $\delta\rho_\perp$  of  $\rho_\perp$  implies that

$$\delta\mathcal{V} + \alpha(z, s) \delta \int \rho_\perp d^2\mathbf{r}_\perp + \beta(z, s) \delta \int \rho_\perp^2 d^2\mathbf{r}_\perp = 0, \quad (20)$$

where  $\alpha(z, s)$  and  $\beta(z, s)$  are Lagrangian multipliers. Taking into account Eq. (12), condition (20) allows us to solve the corresponding ordinary inhomogeneous first-order differential equation for  $V_{contr}$  where  $\rho_\perp$  plays the role of the independent variable and  $z$  and  $s$  are parameters, yielding the following general solution

$$V_{contr}(\mathbf{r}_\perp, z, s) = \frac{h(z, s)}{\rho_\perp(\mathbf{r}_\perp, s)} - \alpha(z, s) - \beta(z, s)\rho_\perp(\mathbf{r}_\perp, s), \quad (21)$$

where  $h(z, s)$  is an arbitrary function. Actually, to ensure the convergence of the integral in the definition of the functional  $\mathcal{V}$ , see Eq. (12), it is easy to see that we must have  $h(z, s) = 0$ . Consequently, the appropriate  $V_{contr}$  satisfying the stationarity condition  $\mathcal{V} = V_0$  is given by

$$V_{contr}(\mathbf{r}_\perp, z, s) = \left[ \frac{q_{1D}(s)}{q} - \rho_\perp(\mathbf{r}_\perp, s) \right] \beta(z, s) + V_0, \quad (22)$$

which after the substitution in Eq. (19) gives

$$\left(i \lambda_c \frac{\partial}{\partial s} - \hat{H}_z\right) \psi_z + [q_{1D}(s) - q|\psi_\perp|^2] (|\psi_z|^2 - \beta/q) \psi_z = 0. \quad (23)$$

Now, according to the hypothesis (13), to preserve the  $\mathbf{r}_\perp$ -independence of  $\psi_z$ , Eq. (23) can be satisfied only when

$$\beta(z, s) = q|\psi_z(z, s)|^2, \quad (24)$$

which immediately implies that Eqs. (17) and (18) are satisfied.

## DECOMPOSITION THEOREM 2.

*Let us suppose that  $\psi_z = \psi_z(z, s)$  is the solution of the 1D nonlinear Schrödinger equation (17). Then, the functional  $\mathcal{V}$  given by (12) and conditioned by the constraints (15), and (16), is stationary (with respect to variations  $\delta\rho_\perp$  of  $\rho_\perp$ ),  $\mathcal{V} = V_0$  if, and only if,  $\psi_\perp = \psi_\perp(\mathbf{r}_\perp, s)$  is the solution of the 2D linear Schrödinger equation (14).*

To prove this proposition, we observe that since  $\psi_z(z, s)$  satisfies Eq. (17), Eq. (10) becomes

$$\left(i \lambda_c \frac{\partial}{\partial s} - \hat{H}_\perp\right) \psi_\perp + [(q_{1D}(s) - q|\psi_\perp|^2) |\psi_z|^2 + (V_0 - V_{contr})] \psi_\perp = 0. \quad (25)$$

By multiplying the latter on the left by  $\psi_\perp^*$  and integrating over all the transverse plane, we easily obtain

$$\int \psi_\perp^* \left(i \lambda_c \frac{\partial}{\partial s} - \hat{H}_\perp\right) \psi_\perp d^2 \mathbf{r}_\perp + V_0 - \mathcal{V}[\rho_\perp; z, s] = 0, \quad (26)$$

where constraints (15) and (16) have been used. Consequently, if  $\psi_\perp$  satisfies Eq. (14), then  $\mathcal{V}$  is a stationary functional with the value  $\mathcal{V} = V_0$ , conditioned by (15) and (16). Conversely, the assumed stationarity of  $\mathcal{V}$  implies that the functional form of  $V_{contr}$  with respect to  $\mathbf{r}_\perp$ ,  $z$  and  $s$  is given by Eq. (22), which substituted in Eq. (25) gives

$$\left(i \lambda_c \frac{\partial}{\partial s} - \hat{H}_\perp\right) \psi_\perp + (q_{1D}(s) - q|\psi_\perp|^2) (|\psi_z|^2 - \beta(z, s)/q) = 0. \quad (27)$$

However, if  $\beta(z, s)/q \neq |\psi_z(z, s)|^2$ , then  $\psi_z$  would be also function of  $\mathbf{r}_\perp$  which would contradict the assumption  $\psi_z = \psi_z(z, s)$ . It follows that  $\beta(z, s)/q = |\psi_z(z, s)|^2$  and, in turn, that Eq. (14) is satisfied.

The results presented above allow us to draw the following conclusion.

*If  $\psi_\perp(\mathbf{r}_\perp, s)$  and  $\psi_z(z, s)$ , are two complex functions which are exact solutions of the 2D linear Schrödinger equation (14) and the 1D nonlinear Schrödinger equation (17), respectively, provided that  $V_{contr}$  is given by Eq. (18), the function  $\psi(\mathbf{r}, s) = \psi_\perp(\mathbf{r}_\perp, s) \psi_z(z, s)$  is the exact solution of the controlled 3D Gross-Pitaevskii equation (3).*

Of course, the inverse is not necessarily true. In fact, it is easy to see that, in principle, it is not true that an arbitrary solution of Eq. (3) can be expressed as the product of two wave functions  $\psi_\perp(\mathbf{r}_\perp, s)$  and  $\psi_z(z, s)$  that obey the Eqs. (14) and (17), respectively. In other words, we can decompose the controlled 3D GPE (3) into the system of equations (14), (17) and (18) only for the subset of its solutions of the type (4). However, using such a decomposition we are able to solve Eq. (3) and to obtain a wide spectrum of exact solutions of the type (4).

#### IV. CONCLUSIONS AND REMARKS

In this paper, we have presented some mathematical properties of the controlled 3D GPE (3). After formulating and proving two decomposition theorems, we have found the



mathematical conditions that make possible the construction of the solution in a factorized form, i.e.  $\psi(\mathbf{r}, s) = \psi_{\perp}(\mathbf{r}_{\perp}, s) \psi_z(z, s)$ , where  $\psi_{\perp}(\mathbf{r}_{\perp}, s)$  and  $\psi_z(z, s)$  satisfy the 2D linear Schrödinger equation ( $i \lambda_c \partial \psi_{\perp} / \partial s = \hat{H}_{\perp} \psi_{\perp}$ ) and the nonlinear controlled nonlinear Schrödinger equation ( $i \lambda_c \partial \psi_z / \partial s = \hat{H}_z \psi_z$ ), respectively. The results presented here improve the formulation of the recently proposed Controlling Potential Method [10, 11].

It is worthy observing that the set of equations (14), (17) and (18) opens up the possibility to find the controlled solutions of the type (4) which exhibit the quantum character in the transverse part (superposition principle with consequent interference effects) and the classical character in the longitudinal part (due to the nonlinearity of the 1D nonlinear Schrödinger equation), although the entire solution of the controlled 3D GPE is nonlinear and, therefore, has a classical character. By means of suitable controlling and trapping potentials, this possibility would allow, for instance, for a very stable soliton-like longitudinal profile of the BEC whose transverse profile would have a quantum character as a result of the quantum interference at the macroscopic level.

Note that, when  $\psi_{\perp}$  satisfies Eq. (14), according to definition (12),  $\mathcal{V}$  represents the average of  $V_{contr}$  in the transverse plane. The value of this average corresponds to the arbitrary constant  $V_0$ . Without loss of generality, we put  $V_0 = 0$ , viz.

$$\int d^2 \vec{r}_{\perp} \psi_{\perp}^* V_{contr} \psi_{\perp} = 0. \quad (28)$$

This way, among all possible choices of  $V_{contr}$ , we adopt the one which does not change the mean energy of the system (note that the average of the Hamiltonian operator in Eq. (3) is the same with or without  $V_{contr}$ ) and thus minimizes the effects introduced by our control operation.

In our forthcoming papers, we will use the method developed in the present paper to solve exactly the 3D controlled GPE with a 3D parabolic potential trap. We find the controlled envelope solutions in the form of localized as well as periodic structures for which suitable stability analysis is performed.

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